COBE Satellite Measurement, Hyperspheres, Superstrings and the Dimension of Spacetime

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Abstract — The first part of the paper attempts to establish connections between hypersphere backing in infinite dimensions, the expectation value of $\dim \mathcal{E}^{(n)}$ spacetime and the COBE measurement of the microwave background radiation.

One of the main results reported here is that the mean sphere in $S^{n-1}$ spans a four dimensional manifold and is thus equal to the expectation value of the topological dimension of $\mathcal{E}^{(n)}$.

In the second part we introduce within a general theory, a probabilistic justification for a compactification which reduces an infinite dimensional spacetime $\mathcal{E}^{(n)}$ $(n = \infty)$ to a four dimensional one $(D_4 = n = 4)$. The effective Hausdorff dimension of this space is given by $\langle \dim_H \mathcal{E}^{(n)} \rangle = d_H = 4 + \phi^2$ where $\phi^2 = 1/2\phi^2$ is a PV number and $\phi = (\sqrt{5} - 1)/2$ is the Golden Mean.

The derivation makes use of various results from knot theory, four manifolds, noncommutative geometry, quasi periodic tiling and Fredholm operators. In addition some relevant analogies between $\mathcal{E}^{(n)}$, statistical mechanics and Jones polynomials are drawn. This allows a better insight into the nature of the proposed compactification, the associated $\mathcal{E}^{(n)}$ space and the Pisot-Vijayaraghavan number $1/\phi^2 = 4.256067977$ representing its dimension. This dimension is in turn shown to be capable of a natural interpretation in terms of Jones' knot invariant and the signature of four manifolds. This brings the work near to the context of Witten and Donaldson topological quantum field theory.

INTRODUCTION

With supergravity, superstrings and supermembranes [1], it is clear that the issue of dimensionality has become of crucial importance, yet none of these theories have come anywhere near to giving a credible explanation let alone a mathematical derivation of why the world appears to us to be $3 + 1$ dimensional or why the alleged extra dimensions should spontaneously compactify [1-5].

There have of course been numerous attempts in the not so distant past to make our familiar $3 + 1$ space plausible using various heuristic arguments. These could be divided into two main groups, physical and mathematical. An example for a heuristic physical explanation would be H. Weyl’s argument that Maxwell’s theory exists only in $3 + 1$ space time dimensions. The argument used by P. Ehrenfest to reason, based on stability considerations, that only in $3 + 1$ could we have stable atoms and consequently chemistry and life belong to the same category. An example for mathematical arguments would be those related to the property of Riemannian spaces and the number of independent components of the Weyl tensor where it can be shown that only the non-trivial conformal structure of three dimensions leads to the properties of Einstein’s gravity in four dimensional space time [3, 4].

A relatively more recent mathematical argument which is somewhat more pertinent to the spirit of our present derivation is that of Fake R³. This important result in topology implies that
while for \( n \neq 4 \) there is always a corresponding unique differentiable structure, in \( n = 4 \) there are uncountably many such structures [6, 7].

In several previous publications we have shown how we could derive the four dimensionality of spacetime based on pure geometrical and set theoretical considerations [7–12]. To do that we had to make one main assumption namely that micro space time is a hierarchical infinite dimensional transfinite set resembling the Ord-Nottale fractal spacetime or the so called Bethe lattice model of branched polymers [7–12]. Proceeding that way it was found that this space which is called \( \delta^{(*)} \) although strictly speaking infinite dimensional, has a finite topological and Hausdorff dimension equal to \( n = 4 \) and \( \langle d_\gamma \rangle = 4.236 \) respectively. In addition the expectation value of \( n \) is \( \sim \langle n \rangle = 4 + \phi^3 = \langle d_\gamma \rangle \) where \( \phi = (\sqrt{5} - 1)/2 \) is the Golden Mean.

Consequently the effective core of \( \delta^{(*)} \) is basically a four dimensional ‘fractal’ manifold. In fact it can easily be shown that \( \delta^{(*)} \) is a random set and represents a four dimensional version of the one dimensional random Cantor set of Mauldin-Williams theorem [10–12].

Noting the remarkable fractional representation of \( \sim \langle n \rangle = \langle d_\gamma \rangle = 4 + \phi^3 = 1/\phi^3 = 4 + (\bar{4}) \) we can see that \( \delta^{(*)} \) seems to suggest an imaginative picture of a four dimensional space containing another smaller four dimensional space and so on \( \text{ad infinitum} \) in selfsimilar fractal fashion. In addition the entire fractal space is imbedded in a space with a topological dimension \( n = 4 \). The first part of the present analysis attempts to develop an analogous picture to \( \delta^{(*)} \) however this time based on the classical result of the volume of \( n \) dimensional hypersphere. The space which we will attempt to construct will be referred to as \( S^{(*)} \) space. We will be showing that \( S^{(*)} \) has many features in common with \( \delta^{(*)} \) and few even more unexpected properties. However it will be shown here that \( S^{(*)} \) may have a more direct connection to physics as will be argued using the results of the COBE satellite measurement of the microwave background radiation [13].

It has been noted frequently for instance, by M. Duff in [1] that superstrings have as yet no answer to the question of why our universe appears to be four dimensional, let alone why it appears to have a signature [1; 3].

In the main part of the present work we intend to give an answer to this question. In the course of doing that, we will be utilizing and also discovering various analogies and some nontrivial relations between our probabilistic approach to ‘compactification’ and several other branches of current research in pure and applied mathematics, in particular knot theory [19] and noncommutative geometry [20, 21].

Our main thesis is that the dimensionality of spacetime, as occasionally speculated in the past by some notable scientists, is a derivable property akin to temperature [31]. In fact we will be showing that the apparent dimensionality of our every day space time \( D_T = 4 \) is derivable from more primitive assumptions and follow from the same law of statistical distribution used by M. Planck to derive his well known formula for black body radiation. This formula we may recall was at least historically speaking the beginning of quantum physics. Naturally, to undertake such analysis we will have to introduce some radical deviation from our classical notion of spacetime. The most decisive point in that respects is the introduction of randomness and scale invariance to the very concept of spacetime geometry which inturn shows that spacetime loses it’s smoothness when we sharpen the resolution of observation as is essential for the micro spacetime of quantum physics. This space time which we refer to as Cantor space time \( \delta^{(*)} \) for obvious reasons have some remarkable properties [21–25]. First, it is an infinite dimensional hierarchical and random geometrical manifold with infinite numbers of equivalent pathes (connections) between any two points. Second, any so called point in this space will always reveal a structure on a close examination, so that strictly speaking the concept point does not exist in \( \delta^{(*)} \) which is a resolution dependent zoom space. It then turned out that \( \delta^{(*)} \) is basically a form of noncommutative geometry. The simplest and best studied example for such a geometry would be the famous Penrose tiling. It is therefore not surprising that Penrose tiling was presented from the very beginning as a low dimensional example for \( \delta^{(*)} \) as well as NCG which obeys a noncommutative C*-Algebra [20].
Another point of importance is the connection between the basic formulas defining $\mathcal{E}^{(\infty)}$ and Jones knot polynomial $V_L$ [21](e). In fact it turned out that a fundamental upperbound for $V_L$ is given by a formula which is a special form of the basic bijection formula of $\mathcal{E}^{(\infty)}$ and that the well known Hausdorff dimension of a quantum path $d_H = 2$ may be derived from the properties of knots in $\mathbb{R}^4$. Finally we will show that the set of Penrose tiling and quasi crystallography provides a unique space where the classical and quantum description of spacetime meet [21](c).

Adding the results of our $S^{(\infty)}$ space analysis to that found in analysing the $\mathcal{E}^{(\infty)}$ space in connection with knot theory, Noncommutative geometry and Penrose tiling, we feel that our main thesis, namely that whatever is generally perceived as strangely nonclassical quantum mechanical is in fact reducable to the nonclassical Cantorian-fractal geometry of spacetime itself is reinforced and justified.

1. PART I: THE SPACES $S^{(\infty)}$ AND $\mathcal{E}^{(\infty)}$

1.1. Remarks to the mathematics of hyperspaces

To gain easy access to the present analysis it is essential to become first familiar with some basic facts related to the mathematics of hyperspaces. Our approach to the subject is informal and differs from the customary one usually encountered in the relevant literature on geometric combinatorics [17].

1.1.1. Metric and hypersurface of 3-geometry. We start from the well known metric of hypersurface with positive curvature [18]

$$d\sigma^2 = r^2[d\chi + \sin \chi (d\theta^2 + \sin \theta d\phi^2)]$$  \hspace{1cm} (1.1)

To visualize the 3-geometry involved, one imagines embedding the metric in four dimensional Euclidian space. This is done by setting

$$\begin{cases}
\omega = r \cos \chi; \\
Z = r \sin \chi \cos \theta; \\
x = r \sin \chi \sin \theta \cos \phi; \\
y = r \sin \chi \sin \theta \sin \phi
\end{cases}$$  \hspace{1cm} (1.2)

with

$$0 \leq \chi \leq \pi, \hspace{0.5cm} 0 \leq \theta \leq \pi, \hspace{0.5cm} 0 \leq \phi \leq 2\pi$$

the volume of the hypersurface becomes

$$V = \int_S (r \sin \chi)(r \sin \chi \sin \theta d\phi)$$

$$= \int_0^{2\pi} 4\pi^2 r^3 \sin \chi(r d\chi) = 2\pi^2 r^3 = 2\pi^2$$  \hspace{1cm} (1.3)

where we have set $r = 1$ to obtain the unit sphere. We recall for later use that the expression for the classical sphere in 3D Euclidian space is $V = 4\pi r^2 = 4\pi$

1.1.2. Generalization of the notion of a sphere. It is useful to introduce the following notation and terminology:

We denote the classical sphere as two spheres, thus referring to the dimension of the surface so that we may write $\mathcal{A} S^{3-1} = \mathcal{A}(S^3)$ to mean the surface area of the classical sphere. On the
other hand when we are referring to the volume $V$ of the sphere,* then we write $V(S^{(n)})$ where $n = 3$ is the dimension of the space in which the sphere is embedded. Let us generalize now the notion of sphere to higher and lower Euclidean space than the classical 3 dimensional one. The first generalization we have already encountered by determining the surface area of a three dimensional hypersphere in four dimensions. However we could regard the circle as a one dimensional sphere in two dimensional space. The area of the circle $\pi r^2$ would correspond then to the volume of this one sphere while the length of the circumference $2\pi r$ will correspond to the surface area. In other words we have for $n = 2$ a unit sphere with

$$A(S^{(n-1)}) = A(S^{(1)}) = 2\pi r = 2\pi$$

and

$$V(S^{(n)}) = V(S^{(2)}) = \pi r^2 = \pi$$

For $n = 3$ we have a similar situation namely

$$A(S^{(n-1)}) = A(S^{(2)}) = 4\pi r^2 = 4\pi$$

and

$$V(S^{(n)}) = V(S^{(3)}) = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi$$

we see there is a pattern here because

$$A(S^{(1)}) = 2V(S^{(2)})$$

and

$$A(S^{(2)}) = 3V(S^{(3)})$$

so one may guess that in four dimensions

$$A(S^{(3)}) = 4V(S^{(4)})$$

Now we have found earlier on from our analysis of the hypersurface with positive curvature that $V = 2\pi^2$. That means $A(S^{(3)}) = 2\pi^2$ and therefore, if the above relation is correct then we must be able to deduce that

$$V(S^{(4)}) = \frac{A(S^{(3)})}{4} = \frac{2\pi^2}{4} = \frac{\pi^2}{2}$$

More generally we should have

$$A(S^{(n-1)}) = nV(S^{(n)}) \tag{1.4}$$

Now in order to verify the correctness of these formulas we need to derive a general formula for determining the volume of $n$ dimensional sphere. This we do next. Let us start by the following improper integral

*A more conventional terminology is to speak of the unit ball rather than sphere when considering the volume.*
\[ \int_{-\infty}^{+\infty} e^{-x^2} \, dx \]  

(1.5)

Generalizing to two directions (dimensions) it is evident that

\[
\left( \int_{-\infty}^{+\infty} e^{-x^2} \, dx \right)^2 = \left( \int_{-\infty}^{+\infty} e^{-x^2} \, dx \right) \left( \int_{-\infty}^{+\infty} e^{-y^2} \, dy \right)
\]

\[ = \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} \, dx \, dy \]

Setting \( r = \sqrt{x^2 + y^2} \) one finds

\[
\left( \int_{-\infty}^{+\infty} e^{-x^2} \, dx \right)^2 = \int_{0}^{2\pi} \int_{0}^{\infty} e^{-r^2} r \, dr \, d\phi
\]

\[ = 2\pi \left[ -\frac{1}{2} e^{-r^2} \right]_0^\infty \]

\[ = \pi \]

That means

\[ \int_{-\infty}^{+\infty} e^{-x^2} \, dx = \sqrt{\pi} \]  

(1.6)

Generalizing to \( n \) dimensions we obtain

\[
(n^{(n/2)}) = \left( \int_{-\infty}^{+\infty} e^{-x^2} \, dx \right)^n
\]

\[ = \int_{-\infty}^{+\infty} e^{-(y_1^2+y_2^2+\cdots+y_n^2)} \, dx_1 \, dx_2 \cdots \, dx_n
\]

\[ = \int_{A(S^{n-1})} \int_{0}^{+\infty} e^{-r^2} r^{n-1} \, dr \, dA \]  

(1.7)

and since \( A(S^{n-1}) = n \, V(S^n) \) one finds that

\[
(n^{(n/2)}) = nV(S^n) \int_{0}^{+\infty} (e^{-r^2})(r^{n-1}) \, dr
\]

Setting \( y = r^2 \) one obtains

\[
(n^{(n/2)}) = \frac{n}{2} V(S^n) \int_{0}^{+\infty} e^{-y^{(n/2)-1}} \, dy
\]

(1.8)

However we know that the gamma function is defined as

\[ \Gamma(n) = \int_{0}^{+\infty} e^{-x^{n-1}} \, dx \]

(1.9)

Therefore we can write that
Table 1. Numerical values for the volume $V(S^n)$ of $n$-dimensional spheres and the density of sphere packing $\eta_{n}$ using $n$-dimensional laminated lattices (see also Fig. 1). The values for $n=24$ correspond to the famous leech lattices. The $\eta_{n}$ values are calculated using eqn. (1.19) while $V(S^n)$ are calculated using eqn. (1.10).

<table>
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<th>$V(S^n)$</th>
<th>$\Delta_{n}$</th>
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$$\pi^{(n,2)} = \frac{n}{2} V(S^n) \Gamma\left(\frac{n}{2}\right)$$

or

$$\pi^{(n,2)} = V(S^n) \Gamma\left(\frac{n}{2} + 1\right)$$

Thus we have the volume of $n$ sphere in terms of the gamma function and the dimensions of the space

$$V(S^n) = \pi^{(n,2)} / \Gamma\left(\frac{n}{2} + 1\right)$$

(1.10)

The numerical evaluation of this formula is displayed in the second column of Table 1. Using this result we can verify immediately that $A(S^0)$ is given correctly by $A(S^0) = 2\pi^2 r^3 = 2\pi^2$ and that $V(S^0)$ is indeed $V(S^0) = A(S^3) / n = (2\pi^2 / 4) = (\pi^2 / 2) = 4.9348$. Naturally the above equation holds for all $n$.

2.1.3. The empty set and $V(S^{(−1)})$. In $\sigma^{(−1)}$ space the truly empty set was found to be associated with $n = −\infty$ and is given by $d^{(−1)}_{\infty} = 0$ where $d^{(\infty)}$ is a Hausdorff dimension. This is of course
different from the classical Menger-Urysohn definition of the empty set which has a dimension $n = -1$. Following Table 1 it is interesting to see that for $n=0$ the volume is non zero, in fact it is unity $V(S^{(0)}) = 1$. Consequently we would like to determine $V(S^{(-1)})$. To do that we start by the elementary fact that the volume of the unit sphere may be determined by integrating over $(n-1)$ dimensional sections, so that we may write

$$V(S^{(n)}) = 2 \int_{0}^{1} V(S^{(n-1)})(1-x^{2})^{\left(\frac{n-1}{2}\right)}dx \quad (1.11)$$

Setting $x = \sin \theta$ and $dx = \cos \theta \ d\theta$ leads to

$$V(S^{(n)}) = 2V(S^{(n-1)}) \int_{0}^{\frac{\pi}{2}} \cos^{n-1}\theta \ d\theta \quad (1.12)$$

For $n = 0$ we have thus

$$V(S^{(-1)}) = \frac{V(S^{(0)})}{2 \int_{0}^{\frac{\pi}{2}} (\cos \theta)^{-1} \ d\theta} \quad (1.13)$$

Noting that $V(S^{(0)}) = 1$ and integrating one finds

$$V(S^{(-1)}) = 1/2[\ln \sec x + \tan x]_{0}^{\frac{\pi}{2}} = 1/2[\ln(\tan \pi/2) - \ln(\tan \pi/4)] = \frac{1}{2\pi} = 0 \quad (1.14)$$

Consequently the ‘empty’ sphere $S^{(-1)}$ has a zero volume. So in case of $n<0$ we have the identity

$$V(S^{(-1)}) = V(S^{(-2)}) = \ldots V(S^{(-\infty)}) = 0 \quad (1.15)$$

### 1.2. The relation between $A$ and $V$ of hyperspheres

We have established the relationship

$$A(S^{(n-1)}) = nV(S^{(n)})$$

It works as we have shown for the circle and the classical sphere. For the circles we have $n=2$. Thus

$$A(S^{(1)}) = 2V(S^{(2)})$$

and since $V(S^{(2)}) = 2\pi$ which is the area of a circle, then we find $A(S^{(1)}) = 2\pi$ which is the length of the circumference of the circle. The conflicting terminology is of course a little confusing at the very beginning. For the classical sphere we have $n=3$. Thus

$$A(S^{(2)}) = A$$

of the sphere $= 3V(S^{(3)})$

and since $V(S^{(3)}) = 4\pi/3$ is the volume of the classical sphere then one finds $A(S^{(2)}) = 3 \cdot \frac{4}{3} \pi = 4\pi$ which is the surface area.
For the hypersurface with positive curvature we have \( n = 4 \) and therefore

\[ A(S^{(4)}) = 4V(S^{(4)}) \]

Now following our Table 1 we have

\[ V(S^{(4)}) = \pi^2/2 \]

Thus

\[ A(S^{(3)}) = 4\pi^2/2 = 2\pi^2 \]

which is the first result obtained earlier.

Next we go further and take \( n = 0 \). Consequently

\[ A(S^{(-1)}) = (0)(V(S^{(0)})) \]

which is always zero. Thus we may write

\[ A(S^{(-1)}) = 0 \]

However we have already established that \( V(S^{(-1)}) = 0 \). Consequently we must have

\[ A(S^{(-1)}) = V(S^{(-1)}) = 0 \]

(1.16)

Let us take \( n = 1 \). Thus

\[ A(S^{(0)}) = (1)(V(S^{(1)})) \]

Since \( V(S^{(1)}) = 2 \), then

\[ A(S^{(0)}) = 2 \]

(1.17)

That means \( A(S^{(0)}) = 2 \) but \( V(S^{(0)}) = 1 \)

This confirms the basic fact that in \( S^{<0} \) we have for \( n < 0 \)

\[ A(S^{(-1)}) = \cdots = A(S^{(-n)}) = V(S^{(-1)}) = \cdots = V(S^{(-n)}) = 0 \]

(1.18)

1.3. The volume \( V \) of \( S^{(0)} \) and packing density \( \Delta_{(0)} \)

Next we evaluate \( V(S^{(0)}) \) using eqn (1.10) as shown in Table 1. We complement this Table with the values for maximum density of lattice sphere backing \( \Delta_{(0)} \) given in [15] by

\[ \Delta_{(0)} = \rho^\nu V(S^{(0)}) = \pi^{n/2}/\rho^{n/2}/\Gamma\left(\frac{n}{2} + 1\right) = \sqrt{\det A} \]

(1.19)

Here \( \rho^\nu \) is taken to be half the minimal distance between lattice points, \( \sqrt{\det A} = M \) and \( M \) is the generator matrix [15].

It may be instructive to compare eqn (1.10) and eqn (1.19) with the gamma distribution function

\[ f(x) = \frac{1}{\Gamma(r)}x^{r-1}e^{-x} \]

(1.20)

which was used to find \( \sim \langle n \rangle = 4 + \phi^3 \) [5]. Notice that in all the three expression we are dividing
The shrinking volumes $V$ of the unit $n$ dimensional hypersphere $S^n$ as the dimension $n$ increases. Notice that for $n=0$ we have $V(S^0)=1$, while for $n\to\infty$ we have $V(S^n)\to 0$. For $n=5$ on the other hand $V(S^5)$ attains its maximal value $V(S^5)=5.263$. Notice also the strong resemblance between this figure and the curve of the COBE measurement shown in Fig. 2. All values of $V(S^n)$ are taken from Table 1 and are calculated using eqn (1.10).

by gamma. In fact in the $V(S^n) - n$ space the curve of the volume versus dimension shown in Fig. 1 resembles the curve of a gamma distribution [13].

Inspecting Table 1 we note the following important facts:

1. The largest unit sphere is $V(S^3)=5.2637$ and for $n\to\infty$ we have $V(S^n)\to 0$.
2. The density for $n=4$ is $\Delta_{44}=0.6168$ which is very close to the Golden Mean. This may indicate a connection to the so called space filling condition [5].
3. For $n=5$ the value of $V(S^3)$ is only slightly larger than 5 so that this space is reminiscent of the condition of space filling [5].
4. The largest product $(n)\otimes (\Delta_{n,4})$ is 2.467 and corresponds to $n=4$. The largest product $V^{(n)}\otimes \Delta_{(n)}$ corresponds also to $n=4$.
5. For $n=3$ we have $V(S^3)\geq n+1=4$ while for $n=0$ we find $V(S^0)=1$.
6. For $n=8$ we $V(S^8)\geq (n/2)=4$
7. For $n=24$ we have $V(S^{24})=\Delta_{243}=0.00193$. This is the famous leech lattice spheres packing.
8. The curve $n - V(S^n)$ of Fig. 1 resembles a gamma distribution. It also has a striking similarity to the COBE curve [13] of microwave background radiation (shown in Fig. 2) to the extent that using appropriate units (x in cycles/centimetres) both curves attain a maximum at exactly 5. Both curves start at a non zero value for $x=0$ and $y$ tends to zero as $x$ tends to infinity. We speculate that there must be a deep relation between $V$ of $n$ dimensional spheres, our space $\delta^{(n)}$ and the thermodynamics of a blackbody. The COBE curve would amount in this case to a possible experimental verification of the infinite dimensionality and hierarchical structure of real micro spacetime.

The preceding observations should also be compared with previous observations reported in [5].

Another important observation apart of $V(S^{24})=\Delta_{243}$ is that $n=5$ is appropriate for fermions because having the largest $V(S^5)$, it may be regarded in a sense to be what is needed for the ‘extra dimension’ of the ‘Cantorian’ Fermions superspace of $\delta^{(n)}$. 
Fig. 2. The variation of the intensity of the microwave background radiation with its frequency as observed by COBE satellite. The observations display a perfect fit with the curve expected from pure heat radiation with temperature of 2.73 K. Notice first the general likeness to the curve of the shrinking spheres of Fig. 1 and second that the maximum of both curves in Figs 1 and 2 lies at the numerical value 5. The correspondence holds, of course, only for the unit used in Fig 2. The resemblance is not a coincidence and goes back to the involvement of the gamma function in both curves (see eqns (1.19) and (1.10)).

It may also be possible using Table 1 to show that $n = 4$ is a mean value of all dimensions. This would reinforce our earlier results for $\mathcal{E}_n^{(x)}$ namely that $n = 4$ and $\langle n \rangle = 4 + \phi^2 = 4 + (4)$. This and the agreement with the COBE curve of Fig. 2 would give a rather strong indication of why it appears as if we were living in 4 dimensional spacetime only, although all modern physical theories seem to require a far larger dimensionality [1–3]. Such analysis is the subject of the next section.

### 1.4. The Cantorian space $\mathcal{E}_n^{(x)}$, the expectation value of the dimension of $\mathcal{E}_n^{(x)}$ and $S_n^{(x)}$

The basic concept and equations of the space $\mathcal{E}_n^{(x)}$ were introduced in some details on numerous previous occasions so that we may be confined here to a short introduction coupled with a summary of the main equations [21–25]. For simplicity we may start with Mauldin-Williams theorem which states that with a probability of one, a randomly constructed triadic Cantor set $S_c^{(0)}$ will have a Hausdorff dimension equal to the Golden Mean.

$$\dim \mathcal{S}_{c}^{(0)} = d_{c}^{(0)} = \phi = (\sqrt{5} - 1)/2$$

(1.21)

The question is now, how could we lift this formula to $n$ dimensions, i.e. what is the Hausdorff dimension of $S_c^{(0)}$. It turned out that, a very simple formula is all that we need. This formula is a generalization of the relationship between the triadic Cantor set $d_c = \ln 2/\ln 3$ and the Sierpinski gasket $d_s = 1/d_t = 1/(\ln 2/\ln 3) = \ln 3/\ln 2$ and is termed the bijection formula [21–25].
COBE satellite measurement, hyperspheres, superstrings and the dimension of spacetime

\[ \dim S^{(n)}_c = d^{(n)}_c = \left( 1 / \dim S^{(0)}_c \right)^{n-1} = \left( 1 / d^{(0)}_c \right)^{n-1} \]  \hfill (1.22)

Consequently, in four dimensions, Mauldin-Williams theorem would be

\[ \dim S^{(4)}_c = (1 / \phi)^{4-1} = \frac{1}{\phi^3} = 4 + \phi^3 = 4.236067977 \]  \hfill (1.23)

where $1/\phi^3$ is known as a Pisot-Vijayvaraghavan number [30].

Next let us construct the $\mathcal{E}^{(\infty)}$ space \textit{ab initio}.

We do that using an infinite number of Cantor sets $S^{(0)}_c$ with all conceivable Hausdorff dimensions in the unit interval. Suppose these sets are all ‘mixed’ together in all possible forms of union and intersections to form one large space made of infinitely many weighted dimensions. Now we ask the following question. What is the expectation value for the dimensionality of this formally infinite dimensional space? To answer this question we need to know the distribution function according to which it was constructed. Assuming a gamma distribution $\Gamma$, the expectation value of this Gaussian distribution is $[21-25]$.

\[ E_{\mathcal{E}}(n) = \langle n \rangle = \mathcal{F} \lambda \]  \hfill (1.24)

Setting the shape factor $\mathcal{F} = 2$ and substituting for the mean value of a Poission distribution of the elementary Cantor sets i.e. $\lambda = \ln(1 / d^{(0)}_c)$ in $E_{\mathcal{E}}(n)$ one finds* \[ \langle \dim \mathcal{E}_{\mathcal{E}}^{(\infty)} \rangle = \langle n \rangle = 2 / \ln(1 / d^{(0)}_c) = \dim \mathcal{H} / \ln(\dim S^{(0)}_c) \]  \hfill (1.25)

where $\mathcal{H}$ is the Hilbert space of Witten theory for two spheres with four points.

Expanding and retaining the linear terms only, we obtain

\[ \sim \langle \dim \mathcal{E}_{\mathcal{E}}^{(\infty)} \rangle \sim \langle n \rangle = (1 + \dim S^{(0)}_c) / (1 - \dim S^{(0)}_c) = \frac{1 + d^{(0)}_c}{1 - d^{(0)}_c} \]  \hfill (1.26)

It was shown in previous work that the last expression is exact within a genuinely discrete and infinite dimensional space [24].

Now to have a space filling it is clear that we must satisfy the following condition [24]

\[ \sim \langle \dim \mathcal{E}_{\mathcal{E}}^{(\infty)} \rangle \sim \langle n \rangle = d^{(0)}_c = \left( 1 / d^{(0)}_c \right)^{n-1} \]  \hfill (1.27)

It is an elementary matter to solve the above equation and show that it can be satisfied if and only if $d^{(0)}_c = \phi = (\sqrt{5} - 1) / 2$ and $n = 4$.

In other words the expectation value for the dimensionality of $\mathcal{E}_{\mathcal{E}}^{(\infty)}$ is identical to Mauldin-Williams theorem in four dimensions. Consequently our $\mathcal{E}_{\mathcal{E}}^{(\infty)}$ space although of infinite dimensions it has an effective finite expectation value for the topological dimension ($n = 4$) and appears therefore as if it were four dimensional. This is a very similar situation to that of a Bethe lattice [21](f). It is not particularly difficult to show that the same space has also an expectation value for the Hausdorff dimension given by

*Equation (1.25) is identical to the equation derived by V. Jones and A. Connes $q = \exp(\pm 2\pi i)$ when solving for $1 / Z = 2 / q$. Consequently $\langle n \rangle = 2 / q$ where $q = 1 / d = (\sqrt{5} + 1) / 2 Z$. Again using the notation of the theory of subfactors it is easily shown that Jones’ $\tau^{-1} = 4 \cos \pi / 2 Z$ is nothing but our eqn (1.26) where one can show that $\tau^{-1} = d^{(0)}_c + 1 / d^{(0)}_c + 2 = 4 + \phi^3$. Thus for $q = 1 / \phi$ we have $\tau^{-1} = \sim \langle n \rangle = 4 + \phi^3$. 


\[\langle \text{dim}_T \mathcal{E}^{(x)} \rangle = \langle \text{dim}_T \mathcal{E}^{(1)} \rangle = \frac{1}{(1 - d_c^{(0)})d_c^{(0)}} = \frac{1}{\phi^3}\]  
(1.28)

The above equation will be shown in paragraph 3 to correspond to eqn (1.21) of the theory of subfactors [20].

We have also given elsewhere some arguments for the fact that the signature of spacetime must be perceived as \(\Sigma = 2\), which we will not repeat here [13].

To sum up, Cantorian spacetime \(\mathcal{E}^{(x)}\) has infinite dimensions but has an effective Hausdorff dimension \(d_H = \langle d_c \rangle \approx 4\) while the topological dimension of the core is exactly 4. We have thus reduced the infinite dimension to a hierarchical four dimensional space. The idea is very different from the original three main methods of compactification used in superstrings namely Toroidal compactification, orbifold compactification and Calabi-Yau compactification. There a finite internal manifold is left over after compactification where the vibration of the strings take place. Such a manifold could be a six dimensional orbifold for instance. A detailed discussion of our approach to this point will be given elsewhere.

It seems therefore that we in the macro world are aware of the 4 dimensions only while the infinite rest are very unlikely to be observed in a sense similar to the extremely tiny compactified dimensions of all types of Kaluza-Klein theories. It is important to note the remarkable continuous fractal representation of \(\mathcal{E}^{(x)}\) namely

\[\sim \langle n \rangle = 4 + \phi^3 = 4 + (4) = 1/4\]  
(1.29)

It suggests an imaginative picture of a 4D fractal universe containing a much smaller 4D fractal universe and so on \textit{ad infinitum}. We may mention that the preceding analysis was inspired by an old proposal due to A. Wheeler. For more details of the analysis and a discussion of it’s implications we refer the reader to [21–25].

Finally applying the so called mean field approximation, it is easily shown that \(\langle d_c \rangle = 4 + \phi^3\) reduces to [5, 7].

\[\langle d_c \rangle = 2\otimes 2 = 2\otimes 2 = 4\]  
(1.30)

Now in the case of the present space of hyperspheres \(S^{(x)}\), it is clear that the corresponding equations are far more complicated than in the case of \(\mathcal{E}^{(x)}\) and a closed form exact expression for the expectations will not be easily obtainable. For this reason we are forced to calculate our mean value of \(\langle \text{dim}_T S^{(x)} \rangle\) numerically, that is to say if it exists at all. Luckily this value, as will be shown shortly, does indeed exist and is easily estimated numerically once the values of \(V(S^{(0)})\) and \(\Lambda_{(0)}\) are determined with sufficient accuracy. In fact a glance at Table 1 will convince us that we can confine our numerical calculations to say \(n = 24\) without affecting the accuracy of the results appreciatively. The numerical procedure to be used here is elementary and amounts to nothing more than finding a centre of gravity. For instance following Table 2 we find in the case of the \(\mathcal{E}^{(x)}\) space that the expectation value of \(n\) is given by

\[\sim \langle \text{dim}_T \mathcal{E}^{(x)} \rangle = \sim \langle n \rangle = \sum_{n=0}^{\infty} \frac{n(n\phi^n)}{n\phi^n} \approx \frac{\sum_{n=0}^{24} n^2\phi^n}{\sum_{n=0}^{24} n\phi^n} \approx 4.262\]  
(1.31)

which is a rather good approximation to the exact result

\[\sim \langle n \rangle = (1 + \phi)/(1 - \phi) = 4.2360679\]  
(1.32)

In a similar way we can determine \(\langle \text{dim}_T S^{(x)} \rangle\) using Table 1 and find that...
Table 1: Weighted moments of the dimensions in a Cantorion space \( \mathcal{S}^{n^2} \). The expression \( n^2 \phi^n \) corresponds to \( \eta_{m,n} \) of the hypersphere space \( \mathcal{S}^{n^2} \) given in Table 0. The expectation value for \( n \) is given by

\[
\langle n \rangle = \sum_{n=0}^{\infty} n^2 \phi^n \sum_{n=0}^{\infty} n \phi^n \approx \frac{17.94427191 \times 10^6}{4.236067977} \approx 4.236067977.
\]

<table>
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<th>( n )</th>
<th>( n\phi^n )</th>
<th>( n^2 \phi^n )</th>
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</tr>
<tr>
<td>( \sum )</td>
<td>4.207129436</td>
<td>17.93314873</td>
</tr>
</tbody>
</table>

\[
\langle \text{dim} \mathcal{S}^{n^2} \rangle = \frac{\sum_{n=0}^{\infty} \eta_{m,n}}{\sum_{n=0}^{\infty} \sum_{m=0}^{n^2} n (V(\mathcal{S}^{m,n})) \Delta_{m,n}} \approx \frac{80.245}{19.726} \approx \frac{4.0678}{1}.
\] (1.33)

If it is not a surprising result to retrieve our familiar four dimensionality, then eqn (1.33) is at least a welcome confirmation of our intuition about the complex fractal-like nature of micro spacetime. It seems that our finite dimensional reality arises somehow through averaging and the complex brain process of visual and tangible perception out of an underlying infinite dimensional surreality.

In fact Tables 1 and 3 could be also used to calculate the mean value \( \langle V(\mathcal{S}^{m,n}) \rangle \). This turned out to be given by

\[
\langle V(\mathcal{S}^{m,n}) \rangle \approx \frac{\sum_{n=0}^{\infty} n V(\mathcal{S}^{m,n}) \Delta_{m,n}}{\sum_{n=0}^{\infty} n \Delta_{m,n}} = \frac{80.245}{21.2759} \approx 3.8 \leq 4
\] (1.34)

This result is also quite interesting because of what may seem at first to be a slight contradiction.
Since \( \langle \text{dim} S^{(n)} \rangle = 4 \) means that the average sphere is four dimensional, then one may have expected following Table 1 that the average volume should be around 5. However we are getting a value around 4. Disregarding the real possibility of rounding and numerical errors the result may be interpreted as follows. Since a volume of about 4 is according to Table 1 the volume of a \( S^3 \) sphere, then we could say that our average 4 dimensional sphere has the volume appearance of a three dimensional sphere. In other words we could use this to justify the division of \( 4D \) into a \( (3 + 1)D \).

Finally we may mention another approximation to eqn (1.34) which is more accurate

\[
\langle V(S^{(n)}) \rangle = \left( \sum_{n=0}^{n=24} V(S^{(n)})(V(S^{(n)})\Delta_{\alpha}) \right) \left/ \sum_{n=0}^{n=24} V(S^{(n)})\Delta_{\alpha} \right.
\]

\[= 78.08753/19.732 \approx 3.957 \approx 4 \quad (1.35)\]

2. PART II: THE \( \theta^{(\alpha)} \) SPACE, KNOT THEORY, NONCOMMUTATIVE GEOMETRY AND RELATED SUBJECTS

2.1. Knot theory and \( \theta^{(\alpha)} \)

A knot is defined as a smooth-embedding of a circle in \( \mathbb{R}^3 \) [19]. An important result in this subject is the discovery of the following index by Jones [20].
It is easily verified that for \( r = 5 \) we find \( J_{\text{ind}} \) to be numerically identical to the Hausdorff dimension of a three dimensional Cantorian space \( \mathcal{S}_c^{(3)} \), when the null set, i.e. the Kernel \( d^{(0)} \) of \( \mathcal{E}^{(0)} \) is taken to be \( d^{(0)} = \phi \).

This is because

\[
J_{\text{ind}} \big|_{r=5} = (2 \cos \pi/5)^2 = 2 + \phi
\]

while the bijection formula [21–25].

\[
\dim_{\mathcal{H}} \mathcal{S}_c^{(n)} = d^{(n)} = (1/d^{(0)})^{n-1}
\]

gives the same result for \( n = 3 \) and \( d^{(0)} = \phi = (\sqrt{5} - 1)/2 \).

Another very important result also found by Jones is the following [26]. If an oriented link \( L \) is a closed \( n \) braid then

\[
|V_L e^{2\pi i r}| \leq (2 \cos \pi/5)^{n-1}; r = 3, 4, \ldots
\]

It is equally easily verified that the right hand side of eqn (2.4) is nothing else but our familiar bijection formula of Cantorian Space \( \mathcal{E}^{(n)} \) and that for \( n = 4 \) and \( r = 5 \) one finds [26, 21](e)

\[
|V_L e^{2\pi i r}| \leq (2 \cos \pi/5)^{3} = (1/d^{(0)})^{3} = \langle \dim_{\mathcal{H}} \mathcal{E}^{(n)} \rangle = 4 + \phi^3
\]

when setting \( d^{(0)} = \phi \). Here we continue to think of \( n \) as the topological dimension of \( \mathcal{E}^{(n)} \) while \( r = K + 2 \) is a parameter where \( K \) can be thought of as the inverse of the Planck constant \( \hbar \).

Equation (2.5) gives thus the expectation value of the dimension of the infinite dimensional hierarchical Cantorian space in this particular case which is [21–25].

\[
\sim \langle n \rangle = \langle d_{r} \rangle = 4 + (3) = 4 + \phi^3 = 2 + \sqrt{5}.
\]

Note that in case of \( r = 5 \) and \( n = 3 \) we have \( |V_L e^{2\pi i r}|_{\mathcal{O}} = J_{\text{ind}} \) where \( 0 \) means evaluated at the corresponding parameters. Note also that for \( n = 4 \) we must have \( 1 \leq |V_L e^{2\pi i r}| \leq 8 \) with \( \frac{1}{2} \leq d^{(0)} \leq 1 \).

### 2.2. Noncommutative geometry (NCG) and Cantorian space time

Noncommutative algebra has for a long time been a basic tool in mathematical physics with the most important application in quantum mechanics. The aim of noncommutative geometry is to extend the idea to algebraic geometry. It then turned out that the spacetime concept of NCG and Cantorian spacetime (CST) have many common features [21](e). For instance one of the topological invariants of a certain very interesting noncommutative or ‘quantum’ space \( X \) which is the dimension group, is a subgroup of \( R \) generated by \( Z \) and the inverse of the Golden Mean \( 1/\phi = 1 + \phi = (\sqrt{5} + 1)/2 \). It followed then that a certain dimension [20]

\[
\dim(e) = \psi(e)
\]

can take only values in the subgroup [20]

\[
Z + \left( \frac{1}{\phi} \right) Z
\]

Following Ref. [20] it can be shown that there is a semigroup \( K_{o} \) such that
\[ K_{\phi}^+(A) = \left\{ (n,m) \in \mathbb{Z}^2, n \left( \frac{1}{\phi} \right) + m \geq 0 \right\} \] (2.9)

It is clear from the last equation that for \( n = 2 \) and \( m = 1 \) we have
\[ n \left( \frac{1}{\phi} \right) + m = 2 \left( \frac{1}{\phi} \right) + 1 = 4 + \phi^3 \] (2.10)

which is identical to what we have obtained for \( d^{(n)} = \langle \dim_{\phi} \mathcal{E}^{(\omega)} \rangle \) in eqns (1.23) and (1.29). Furthermore and following the theory of subfactors and the notation of ref. [20], it can be shown that the index \([M:N]\) of \( N \) in \( M \) [20]
\[ [M:N] = \dim_{\phi}(L^2(M)) \] (2.11)
is also given by
\[ \dim_{\phi}(L^2(M)) = \frac{1}{(1 - \lambda_0) \lambda_0} \] (2.12)
where \( \lambda_0 = \text{Tr} M_{\omega} \). It is clear from the right hand side of (2.12) that this expression is nothing but the formula for the expectation value of the Hausdorff dimension of \( \mathcal{E}^{(\omega)} \) namely \[ \langle \dim_{\phi} \mathcal{E}^{(\omega)} \rangle = \langle \mathcal{H}_\omega \rangle = 1 / [(1 - d_{\text{min}}^{(\omega)})(d_{\text{min}}^{(\omega)})] \] (2.13)

and we just need to set
\[ \lambda_0 = \dim_{\mathcal{E}^{(\omega)}} = d_{\text{min}}^{(\omega)} = \phi = (\sqrt{5} - 1)/2 \] (2.14)
in order to find again our by now familiar number
\[ [M:N] = \langle \mathcal{H}_\omega \rangle = \frac{1}{(1 - \phi) \phi} = \frac{1}{\phi^3} = 4 + \phi^3 \] (2.15)

It should be noted that minimizing \( \langle \mathcal{H}_\omega \rangle \) leads to \( d_{\omega}^{(\min)} = 1/2 \) and
\[ \langle \mathcal{H}_\omega \rangle \text{ min} = 4 = \langle \dim_{\mathcal{E}^{(\omega)}} \rangle \text{ min}. \]

Another interesting correspondence between the formalism of \( \mathcal{E}^{(\omega)} \) and that of NCG is evident from looking at the expression for \( \tilde{t} \) as given on page 59 in [20]. There we see that setting \( q = \phi \) or \( 1/\phi \) in
\[ \tilde{t} = (2 + q + q^{-1})^{-1} \] (2.16)
leads to
\[ \tilde{t} = \phi^3 \] (2.17)
and consequently
\[ 1/\tilde{t} = [M:N] = \langle \mathcal{H}_\omega \rangle = 2 + \sqrt{5} = 4 + \phi^3 \] (2.18)

A similar result is also obtained for Jones polynomial* for \( t = \phi \). That means

*This form of Jones polynomial is given on p. 175 in *The Knot Book* by C. Adams, Freeman, New York (1994).
2.3. Connections to four manifolds

To illustrate the connections between the geometry of four manifolds and \( \mathcal{E}^{(n)} \) consider first the signature \( \psi \) [6]

\[
\psi = b^+ - b^-
\]

where \( b^+ = \text{dim } H^+ \) and \( b^- = \text{dim } H^- \) are the dimensions of the maximal positive and negative subspaces of the form \( H_2 \) respectively [6].

Setting

\[
b^+ = \dim H \ker \mathcal{E}^{(n)} = \phi
\]

and

\[
b^- = \dim H \coker \mathcal{E}^{(n)} = 1 - \phi = \phi^2
\]

one finds from eqn (1.28) that [4][e]

\[
\psi = [b^+ - b^-]_\phi = [b^+ \otimes b^-]_\phi = \phi - \phi^2 = \phi \cdot \phi^2 = \phi^3
\]

Recalling that

\[
\psi = [M:N]^{-1}
\]

then

\[
\dim L^2(M) = [M:N] = 1/\psi = \langle \dim \mathcal{E}^{(n)} \rangle = 4 + \phi^3
\]

This is exactly the same result obtained earlier on [21].

It is worth mentioning here that in [22] it was found that the Fibonacci numbers play a role in the topology of four manifolds. The importance of the PV number, \( 2 + \sqrt{5} = 4.23606 \), seems to extend far beyond quasicrystallography where it found one of it’s first physical application [21][e]. The most important conclusion so far however is that Jones’ knot invariant has a natural interpretation in terms of dimensionality of \( \mathcal{E}^{(n)} \) and the signature of four manifolds.

2.4. Quasicrystallography and \( \mathcal{E}^{(n)} \)

One of the surprises which we have encountered recently is that the expectation value of the effective dimensionality of \( \mathcal{E}^{(n)} \) namely \( \langle n \rangle = 4 + \phi^3 = 2 + \sqrt{5} \) as well as it’s inverse \( \phi^3 \) occurs in connection with the theory of quasicrystallography. In ref. [28] it was found that the Z-module which carries diffraction pattern possesses certain symmetry which is invariant through a group of homotheties [28]. Noting that Shur’s lemma entails under certain conditions that

\[
\Lambda \pi^i + \Lambda' \pi^i
\]

for any real numbers \( \Lambda \) and \( \Lambda' \), one finds that

\[
M = \phi^{-3} \pi^i \phi^3 \pi^i
\]

where \( \pi_i \) and \( \pi^i \) are certain projective matrices [28] and
\[
\phi^{-3} = 4 + \phi^3 = 2 + \sqrt{5} = \langle d \rangle = [M:N]
\] (2.27)

2.5. Penrose universes, NCG and \(\mathcal{E}^{(\infty)}\)

One of the most important results in noncommutative geometry is undoubtedly the conclusion by Connes that the Penrose space \(X\) represents in effect an example of a low dimensional noncommutative space [20].

To appreciate the importance of this result in nonclassical physics we need just recall that while classical mechanics obeys commutative algebra, quantum mechanics in the Heisenberg-Born formalism is manifestly a problem in noncommutative analysis. Thus, Penrose universe is a unique medium where classical and nonclassical physics meets.

The objective of this section is to show the intimate connection between Penrose universe and Cantorian spacetime \(\mathcal{E}^{(\infty)}\), an undertaking which may lead to a resolution of many paradoxes in quantum physics, by reducing them to the nonclassical transfinite nature of the ambient micro spacetime of quantum and subquantum particles. To see how important such an undertaking is as well as the importance of noncommutative geometry it may be sufficient to mention that it is very likely that A. Einstein’s opposition to quantum mechanics stem from the fact that all forms of geometries known to him at that time were commutative. The noncommutative quantum theory of Heisenberg and Bohr may have therefore remained obscure to him because it did not fit into his basically geometric thinking. In other words it may be reasonable to suppose that had Einstein known about the possibility of noncommutative geometry he would most probably modify his attitude towards quantum mechanics [19].

With the experimental discovery of quasicrystals which possesses the supposedly forbidden five fold symmetry, the subject of nonperiodic tiling of space acquired a prominent place in mathematical physics, particularly in the context of the work of R. Penrose and H. Conway on Penrose tiling [20].

The recent work of A. Connes [20] on noncommutative geometry added a new dimension to the importance of nonperiodic tiling by realizing that \(X\) is an example of noncommutative geometry. The starting point of this realization is to look at the case of measured foliations with continuous dimension [20]. The general von Neumann projection of the foliations gives a sort of a random Hilbert space. This is of course reminiscent of Mauldin’s theorem and our Golden Mean theorem [10–15]. It then turned out that the space of Penrose tiling of the plane does indeed give a very clear geometrical intuitive picture of what a leaf of foliation could generally look like. The reason for that is simply the following [20]. When analysing \(X\) using classical tools, it appears to be pathological as observed by Connes. However when we replace the commutative C*-Algebra with a noncommutative C*-Algebra we find that \(X\) is readily analysed [20]. This is a clear indication of the inherently quantum mechanical nature of the space \(X\). In this sense we understand this space as an example of noncommutative geometry of a low dimensional noncommutative space [20].

The preceding reasoning yield then two quantitative results which imply quite unexpected connections of the Penrose space to knot theory and Cantorian spacetime.

The first result is that \(X\) has a natural subfactor which is identical to Johns’ index [20]

\[
J_{\text{ind}} = (2 \cos \pi/5)^2
\] (2.28)

This was discussed earlier on as an important quantity in knot theory [19]. The second point is that \(J_{\text{ind}}\) is itself numerically identical to the Hausdorff dimension of a three dimensional Cantorian space which represents a generalization of Mauldin’s theorem [5] to three dimensions using the so called bijection formula.
These are by no means the only indications that the Penrose universe can be seen as a realization of projection of a three dimensional Cantorian space with

\[ \dim \text{Ker} \mathcal{E}^{(x)} = \dim \langle \phi \text{ of } \mathcal{E}^{(x)} \rangle = \phi \]  

(2.30)

To explain this point we suppose we have a circular region in \( X \) of a diameter \( \rho \). Suppose further that one is transferred to a randomly chosen parallel Penrose universe. Then we ask the following question:

How far do we need to travel from our initial circular region in order to find another circular region which can match the initial one? The answer is that we have to travel a distance

\[ l \sim \frac{1}{4 + \phi^3} \rho = (2.118033989\ldots) \rho \]  

(2.31)

This is the essence of the local isomorphism theorem. This theorem is of course trivial for a periodic pattern, but the Penrose universe is nonperiodic and here lays the first surprise. The second surprise is that \((4 + \phi^3)/2 = 2.118033989\) is exactly equal to half of \(1/\phi^3\) which is the exact expectation value for the dimensionality of the hierarchical Cantorian universe \(\mathcal{E}^{(x)}\). As mentioned earlier on several times this essentially infinite dimensional universe has an effective Hausdorff dimension \(\langle \dim_H \mathcal{E}^{(x)} \rangle = 4 + \phi^3 = 4 + (4)\) and a topological dimension for the effective ‘core’ of exactly \(D_T = n = 4\) as can again be seen immediately from the bijection formula introduced earlier on

\[ \dim S^{(0)} = (1/\dim S^{(0)})^{n-1} \]  

(2.32)

when setting \(n = 4\) and

\[ \dim S^{(0)}_{\mathcal{E}} = d^{(0)} = \phi \]

\[ \dim S^{(4)}_{\mathcal{E}} = d^{(4)} = (1/\dim S^{(4)})^{3} = (1/\phi)^3 = \langle \dim \mathcal{E}^{(x)} \rangle = 4 + \phi^3 \]  

(2.33)

in agreement with our earlier discussion.

It is worth remembering and stressing again that the distribution used to obtain these results is the same distribution used to derive the formula of black body radiation. In this sense dimensionality seems to share indeed some essential features with temperature as anticipated by D. Finkelstein [31].

2.6. Knot theory and the Hausdorff dimension of a quantum path

As mentioned earlier on a knot is by definition a smooth-embedding of a circle in \(\mathbb{R}^3\) [19]. A circle \(C_1\) is of course a \(\dim c_1 = 1\) geometrical object. It can be deformed to a true knot only in a space \(S_1\) where \(\dim S_1 = 3\). Consequently the co-dimension must be

\[ \text{Codim } C_1 = \dim S_1 - \dim C_1 = 3 - 1 = 2 \]  

(2.34)

There are however no ‘real’ knot in \(\mathbb{R}^4\) for which \(\dim S_1 = 4\) because all knots are equivalently trivial and dissolve in this higher dimensional space. It turned out that the only valid way to
generalize the concepts of knot theory to higher dimensions is to keep the Codim = 2 constant. Consequently the geometrical object corresponding to the circle must have a dim C₁ = 2. Therefore in R³ we must have

\[
\text{Codim } C₁ = 4 - 2 = 2
\]

(2.35)

In other words to have a knot in R³ we must have

\[
\text{dim } C₁ = \text{Codim } C₁ = 2
\]

(2.36)

The next point to be reasoned is based on the hypothesis that all particles and all interactions between particles in micro space are manifestation of fractal-like knots in the ‘fabric’ of Cantorian spacetime \( \mathcal{E}^{(c)} \) at an appropriate resolution. Now since \( \mathcal{E}^{(c)} \) is effectively four dimensional by virtue of

\[
\sim \langle \text{dim } \mathcal{E}^{(c)} \rangle = \frac{\text{dim } C₂}{\text{dim } \text{Coker } \mathcal{E}^{(c)}} - 1 = 5.23606 - 1 = 4 + \phi³
\]

\[
\approx 2 \oplus 2 = 2 \oplus 2 = 4
\]

(2.37)

then it follows that we could not have a knot, a particle and consequently a particle path in \( \mathcal{E}^{(c)} \) unless \( \text{dim } C₁ = 2 \) and that means

\[
\langle \text{dim } \mathcal{E}^{(c)} \rangle = \text{dim } C₂ = \text{Codim } C₂ = 2
\]

(2.38)

In this sense a Cantorian spacetime sheet must fall back on itself and thus form an effective four dimensional knot. That is basically why a quasi continuous connection in \( \mathcal{E}^{(c)} \) is essentially a path with a Hausdorff dimension \( \langle \text{dim } \mathcal{E}^{(c)} \rangle = 2 \) [21–25]. In this context two more side remarks are in order. First we recall the Frish-Wasserman-Delbrück conjecture which states that the probability for a randomly embedded circle to be knotted tends to 1 as the length of the circle tends to infinity. For a ‘fractal’ circle, the length of any part is infinite provided the circle appears to be continuous at the corresponding resolution of observation and if the FWD conjecture is correct then a fractal circle is everywhere knotted. Furthermore, FWD implies also that a self-avoiding polygon must be knotted and since in 4 dimension a polygon is naturally self-avoiding this means our ‘circles’ in \( \mathcal{E}^{(c)} \) must be knotted. We will regard all form and particle interactions as a manifestation of these transfinite-fractal knots as indicated earlier on. Finally we may recall that the universe as a whole may be regarded in some speculative models as a knot complement. This concludes our topological justification of \( \langle \text{dim } \mathcal{E}^{(c)} \rangle = 2 \).

2.7. The so called wave-particle duality, NCG and \( \mathcal{E}^{(c)} \)

The aim of the present section is to show that the indistinguishability theorem of Cantorian spaces \( \mathcal{E}^{(c)} \) [21](f)

\[
\hat{\Omega} = \bar{\Omega} = \phi³, \phi = (\sqrt{5} - 1)/2
\]

(2.39)

is conceptually homomorphic to a wave-particle interpretation of the index theorem of Toeplitz operators [29]

\[
\text{Ind}(T(f)) = -\langle [\eta] f^* [T] \rangle
\]

(2.40)

In fact, it can be shown in an elementary fashion that our indistinguishability theorem (equ. 2.39) is derivable from the index theorem of equ. (2.46) which we do next.
Let $f$ be a complex continuous and never vanishing periodic function on $\mathbb{R}$ with a period $2\pi$, then we could write

$$f(x) = e^{inx+\varphi(x)}$$

where $n$ could be interpreted as a winding number:

$$n = \frac{1}{2\pi i} \int_{\mathbb{R}/\mathbb{Z}} \left[f'(x)/f(x)\right] dx$$

(2.41)

(2.42)

for the closed path

$$\gamma = f(T)$$

(2.43)

in the complex plan given by the image

$$T = \mathbb{R}/\mathbb{Z}$$

(2.44)

The winding number can be also expressed as

$$n = \int \frac{1}{2\pi i} dZ = \int \eta$$

(2.45)

where the closed 1-form $\eta$ defines a de Rham cohomology class in the first cohomology group.

Similarly and since $\gamma$ is a closed path, and therefore defines a homology class we have $[\gamma] = f^*[T]$. Consequently using de Rham duality, $n$ can be written as

$$n \Rightarrow \n = -\langle [\eta] f^*[T] = -\langle [\eta] [\gamma]\rangle$$

(2.46)

The right handside of the above equation (2.46) is thus analogous to the Born formula and reflects therefore the homological structure of the Schrödinger wave quantization. Next let us state the wellknown result that the Toeplitz operator $T(f)$ is Fredholm $[29]$. Since a Fredholm operator admits a finite dimensional Kernel and co-Kernel we can state the well known formula

$$\text{Ind}(T) = \text{dim}(\text{Ker}(T)) - \text{dim}(\text{Coker}(T))$$

(2.47)

This formula is clearly analogous to the index of four manifolds when we admit continuous dimensions $[21-25]$.

Now we can state the index theorem

$$\text{Ind}(T(f)) = -\langle [\eta] [\gamma]\rangle = n$$

(2.48)

In words, this means, the index of a Toeplitz operator $T(f)$ is equal to the winding number $n$ of $f$. However since the left handside of (2.47) and (2.48) are a winding number in terms of an operator, it is analogous to the essentially particle picture of the Heisenberg-Born quantization formalism. Consequently we may write

$$n \Rightarrow n = \text{Ind}(T)$$

(2.49)

Next we like to give a derivation of the index theorem of eqn (2.40) using a purely formal analogy between the formalism of the index theorem and that of Cantorian spaces $E^d(x)$. This procedure which holds for knot theory, noncommutative geometry as well as four manifolds $[6]$ will be referred to rather loosely as ‘analagical’ continuation.

Following $[6]$, eqn (2.47) could be rewritten by performing the following replacements
\[
\text{Ind}(T(f)) \rightarrow \psi = b^+ - b^-
\]
(2.50)
\[
\text{dim}(\text{Ker}(T(f))) \rightarrow \text{dim}(\text{Ker}(E^{(\infty)})) = b^+ = \phi
\]
(2.51)
\[
\text{dim}(\text{Coker}(T(f))) \rightarrow \text{dim}(\text{Coker}(E^{(\infty)})) = b^- = \phi^2
\]
(2.52)

where \( \phi = (\sqrt{5} - 1)/2 \) is the Golden Mean.

Consequently one finds
\[
\hat{n} = \text{Ind}(T(f)) = \psi = \phi^3
\]
(2.53)

where \( \phi^3 \) happened to be the inverse of the well known expectation value for the dimension of \( E^{(\infty)} \) namely \( \langle \text{dim}_0 E^{(\infty)} \rangle = 4 + \phi^3 \) which is a PV number.

On the other hand making the following exchanges in eqn (2.46)
\[
[\eta] \rightarrow \zeta - \phi^3
\]
(2.54)

and
\[
[\gamma] \rightarrow \zeta \phi
\]
(2.55)

one finds
\[
\hat{n} = -\langle -\phi^2 | \phi \rangle = \phi^3
\]
(2.56)

Next we invoke the geometric probability interpretation of the Hausdorff dimension of the Kernel and co-Kernel of \( E^{(\infty)} \) as discussed for instance in [10](a). That way we see immediately that \( n \) as given by eqn (2.47) and eqn (2.53) are essentially the addition theorem of independent probability events which can be consistent only with a particle picture.

On the other hand eqn (2.46) and eqn (2.56) are a clear statement of the multiplication theorem which make sense only for extended objects such as a Schrödinger wave and never for a particle. In other words
\[
\hat{n} = \hat{n}
\]
(2.57)

is entirely consistent with
\[
\hat{\Omega} = \hat{\Omega}
\]
(2.58)

as well as the wave-particle duality interpretation of the index theorem [21](b)[29].

### 3. DISCUSSION AND CONCLUDING REMARKS

The preceding analysis, interesting as it may be, would, in our view, account to little more than a numerological curiosity if it were not for two things. First, it is a true fact that we live in 3 + 1 dimensions, although these dimensions seem occasionally to fail in explaining certain experimental results at least in particle physics. Second, the similarity between the curves of
Fig. 1, 3–6 and Fig. 2 of the COBE measurement is truly intriguing to say the least. We feel that if there were ever serious thoughts to regard spacetime and its dimensionality as a process, then this similarity between the basically thermodynamical curve of Fig. 2 and the geometry of \( n \) dimensional spaces of Fig. 1 and 3–6 must be the best argument to use. In fact it may well be that we have the following admittedly unconventional alternatives. Either the COBE curve looks

\[
\frac{\phi^n}{n^2} \propto \frac{1}{n^{\phi}}
\]

or

\[
\phi^n = \frac{1}{n^{\phi}}
\]

\( \phi \) is the Golden Mean.

Fig. 3. The gamma distribution of \( n^2 \phi^n \) where \( n \) is the dimension of the Cantorian space \( \delta^{a+b} \) and \( \phi \) is the Golden Mean (see eqns (1.31) and (1.32)). Notice the similarity to the curve of the shrinking hyperspheres of Fig. 1 as well as the COBE curve of Fig. 2. However, unlike Figs 1 and 2 the curve starts at \( x = y = 0 \) and has a maximum at \( x = n = 4 \) rather than at 5. All values of \( n^2 \phi^n \) are taken from Table 2.

Fig. 4. The weighted \( V(S^n) \) as a function of \( n \) used in calculating the mean value of \( V(S^n) \) in \( S^{a+b} \) (see eqn (1.35)). Note the similarity to the corresponding Fig. 3.
Fig. 5. The curve $\eta_{(n)} - n$ also resembles Figs. 1–4. The values are taken from Table 1 and used to estimate the mean dimension $\langle \dim S_{(\omega)} \rangle$ in $S^{(\omega)}$ (see eqn (1.33)).

Fig. 6. The curve $k_{(n)} - n$ is used to determine the mean value of $n$ in $S^{(\omega)}$ using the expression $\langle n \rangle = \sum \frac{\eta_{(n)}}{k_{(n)}}$, as given by eqn (1.33).

the way it looks because the geometry of micro spacetime forces it to look like that; or micro spacetime looks as it does because it is ultimately the creation of a thermodynamics-like process. In conclusion we may reiterate the words of A. Wheeler [16] which encapsulates the quintessence of our analysis of the $S^{(\omega)}$ and $S^{(\omega)}$ spaces:

Recall the notion of a Borel set. Loosely speaking a Borel set is a collection of points ‘bucket of dust’ which has not yet been assembled into a manifold of any particular dimensionality.
Recalling the universal way of the quantum principle one can imagine probability amplitudes for the points in a Borel set to be assembled into points with this, that and the other dimensionality.

More conditions have to be imposed on a given number of points-as to which has which for a nearest neighbour-when the points are put together in a five-dimensional array than when these same points are arranged in a two dimensional pattern. Thus one can think of each dimensionality as having a much higher statistical weight than the next higher dimensionality.

On the other hand, for manifolds with one, two and three dimensions the geometry is too rudimentary, one can suppose, to give anything interesting. Thus Einstein’s field equations applied to manifold of dimensionality so low, demand flat space, only when the dimensionality is as high as four do really interesting possibilities arise. Can four therefore be considered to be the unique dimensionality which is at the same time high enough to give any real physics and yet low enough to have great statistical weight?

We hope that our result for the expectation value of the dimensionality of both, the dust-like $\varepsilon^{(\kappa)}$ space and the sphere-like $S^{(\kappa)}$ space

$$\sim \langle \dim \varepsilon^{(\kappa)} \rangle \simeq \langle \dim S^{(\kappa)} \rangle \simeq 4$$

has confirmed Wheeler’s idea to a reasonable extent. To be able to make stronger statements we must first have an exact solution to the expectation values of $S^{(\kappa)}$.

In the second part of this paper and starting with a transfinite hierarchical spacetime $\varepsilon^{(\kappa)}$ of infinite dimensions and following some ideas due to A. Wheeler, D. Finkelstein and C. von Weizsäcker, we derive a finite expectation value and an effective Hausdorff dimension for $\varepsilon^{(\kappa)}$ using a gamma distribution. This is the same distribution used to derive the Maxwell velocity distribution law as well as Planck black body radiation formula. This may be seen as an indication for a conceptual link between temperature and dimensions. The transformation $\varepsilon^{(\kappa)} \simeq \varepsilon^{(\kappa)}$ can then be used as a basis for developing a form of transfinite hierarchical superstrings theory which is closely related to knot theory, non-commutative geometry and quasi periodic tiling. The hierarchical dimensional formula $\langle d \rangle = 4 + (4 \approx 4.23 \ldots)$ is obtained for $\varepsilon^{(\kappa)}$ without the need of suppressing any terms such as the lovelace term $1-(D-2)/24$ or invoking supersymmetry and it allows for a wide spectrum of possible quantized ‘vibration’ on a finer and finer ‘dimensional’ scales.

In addition the relativistic demand on a minimum string’s world surface can be met in our model in an elementary fashion by minimizing $\langle d \rangle$ as given by eqn (2.21) and eqn (2.22). The result follows then from Diff $\langle d \rangle = 0$ to $d^{(0)} = 1/2$ and consequently $\langle d \rangle_{\text{min}} = 4$ in a natural way.

Important insight into the meaning of the connection between topological quantum field theory, knots and $\varepsilon^{(\kappa)}$ may be gained by contemplating the meaning of $n$ in both eqn (2.2) for $\dim S^{(n)}$ and eqn (2.13) for $|V_{\epsilon} e^{2\pi i \epsilon}|$. In the first $n$ clearly denotes topological dimension. In the second the meaning of $n$ is slightly more involved. An $n$ braid is a braid group on $n$ strands $B_n$. This $B_n$ can be defined formally as a fundamental group in configuration space $C_n$ of $n$ distinct points. The braid can then be viewed as the spacetime graph of motion along a closed connection in $C_n$ and that establish the analogy to our $\varepsilon^{(\kappa)}$ spacetime. The next step is to look at the classical limit of the corresponding TQF theory by letting $\hbar \rightarrow 0$ in the usual way. Now we know that $r = k + 2$ and that the level of the theory, $k$, plays the same role of $1/\hbar$. Consequently for $\hbar \rightarrow 0$ we have $K \rightarrow \infty$ and thus $r \rightarrow \infty$. This means $\cos \pi r$ goes to unity and we are left with $|V_{\epsilon} e^{2\pi i \epsilon}| \leq (2)^{r-1}$. This means for a space behaviour ($n = 3$) one finds $|V_{\epsilon} e^{2\pi i \epsilon}| \leq d^{(3)} = 4$ which is our classical spacetime dimension indicating that one dimension ($4 - 3 = 1$) will remain invisible in the three dimensional space giving rise to $n = 3 + 1$ spacetime. It is also clear that $2 \cos \pi r = 2$ corresponds in the bijection formula to $\langle d^{(0)} \rangle = 1/2$. In turn for $d^{(0)} = 1/2$ we have $\sim \langle n \rangle = 3$ and
Table 4. Page 462 of “The Curves of Life” by T.A. Cook first published in 1914 showing the remarkable Table of \( \phi \) scale. With the help of this Table Cook analysed Master Works of painting by S. Botticelli (Venus), Franz Hals “The Laughing Cavalier” and Turner (Ulysses deriving polyphemas). The Table is readily reproduced using the bijection formula for \( d^0 = 1/\phi^1 = (\sqrt{5}-1)/2 \). Note that \( \phi^2 \) and \( \phi^1 \) are identical to Johns index and \( \langle \dim_{\text{Hofstadter}} \rangle \) respectively.

\[
\begin{align*}
\phi^1 & : 1.618 & \phi^1 & : 1.618 \\
\phi^0 & : 1.000 & \phi^2 & : 2.618 \\
\phi^1 & : 1.618 & \phi^1 & : 4.236 \\
\phi^0 & : 2.618 & \phi^4 & : 6.854 \\
\phi^1 & : 4.236 & \phi^1 & : 11.090 \\
\phi^3 & : 6.854 & \phi^1 & : 17.944 \\
\phi^2 & : 11.090 & \phi^1 & : 29.034 \\
\phi^2 & : 17.944 & \phi^1 & : 46.979 \\
\phi^3 & : 29.034 & \phi^1 & : 76.013
\end{align*}
\]

\( \langle d, \rangle = 4 \) which reinforces our conclusion of why classical spacetime is 3 + 1 rather than simply 4 dimensional.

It may be quite amazing to note that the rather interesting numbers of Johns’ index \( (2 + \phi) \) and the exact dimension of \( \sigma^{(x)} = (4 + \phi) \) were included in a book published as long ago as 1914 the relevant page of it is displayed in its original form in Table 4.

REFERENCES

COBE satellite measurement, hyperspheres, superstrings and the dimension of spacetime